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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A FORWARD BACKWARD HEAT EQUATION

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ABSTRACT. Let ϕ be a piecewise linear function which satisfies the condition $s\phi(s) \geqslant cs^2$, $c > 0$, $s \in \mathbf{R}$, and which is monotone decreasing on an interval $(a, b) \subset \mathbf{R}_+$. It is shown that for $f \in C^2[0,1]$, with $\max f' > a$, there exists a $T > 0$ such that the initial boundary value problem

$$u_t = \phi(u_x)_x, \quad u_x(0,t) = u_x(1,t) = 0, \quad u(\cdot,0) = f,$$

has infinitely many solutions u satisfying $\|u\|_\alpha, \|u_x\|_\infty, \|u_t\|_2 \leqslant c(f,\phi)$ on $[0,1] \times [0,T]$.

0. Introduction. Consider the initial boundary value problem

$$\begin{aligned} &u_t = \phi(u_x)_x, \qquad (x,t) \in [0,1] \times [0,T], \\ (1) \qquad &u_x(0,t) = u_x(1,t) = 0, \qquad t \in [0,T], \\ &u(x,0) = f(x), \qquad x \in [0,1]. \end{aligned}$$

If ϕ is strictly monotone increasing with $\phi' \geqslant c > 0$, (1) has a unique solution which is, roughly speaking, as smooth as the function ϕ. On the other hand, if $\phi' \leqslant -c < 0$, (1) is a 'backward' parabolic equation and, because of the smoothing effect, may have a solution only for special initial values.

In nonlinear diffusion, for which equation (1) is a simple model in one space dimension, ϕ need not be monotone increasing. The Clausius-Duhem inequality [**D**, p. 79] in one space dimension merely implies that the graph of ϕ lies in the first and third quadrant. An additional, physically reasonable hypothesis regarding ϕ is the coercivity condition

$$\text{(c)} \qquad s\phi(s) \geqslant cs^2, \qquad c > 0.$$

This assumption allows ϕ to have monotone decreasing parts (e.g. the model cubic $\phi(s) = \frac{1}{3}s^3 - \frac{3}{2}s^2 + 2s$).

A natural and interesting question is whether problem (1) has a solution if the usual assumption $\phi' > 0$ is replaced by the weaker coercivity condition (c). Under this hypothesis J. Bona, J. Nohel and L. Wahlbin [**BNW, HN**] obtained several a

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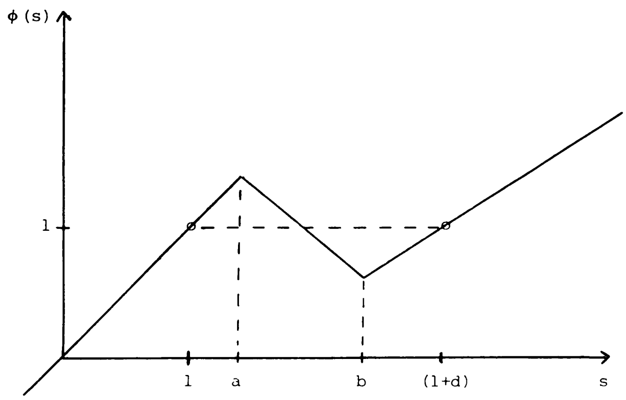
priori estimates for problem (1). Assuming the existence of a solution u in $W_2^1([0,1]\times[0,T])$ they showed, e.g., that

$$\|u(\cdot,T)\|^2_{2,[0,1]}+2c\|u_x\|^2_{2,[0,1]\times[0,T]}\leqslant\|f\|^2_{2,[0,1]},$$
$$\|u_x\|_{\infty,[0,1]\times[0,T]}\leqslant c^{-1}\|\phi(f')\|_{\infty,[0,1]}.$$

For smooth solutions they also proved a maximum principle for u_x. Using Galerkin approximations these estimates almost yield the existence of weak solutions. The difficulty, which prevents completing an existence proof, is that the map $u_x\to\phi(u_x)$ is not weakly continuous if ϕ is not monotone.

The above estimates and J. Nohel's continuing optimism concerning the existence of weak solutions motivated our investigations. We study the simplest case where ϕ is a typical nonmonotone piecewise linear function satisfying (c). We shall assume that ϕ is of the form

(2)



i.e. $\phi(s)=m_1s-(m_1+m_2)(s-a)_+ +(m_2+m_3)(s-b)_+$, $m_j>0$, $\phi(b)>0$.

Clearly, if the initial function f satisfies $\max f'\leqslant a$, then by the maximum principle any solution of the equation $u_t=m_1u_{xx}$ solves problem (1).

If, however, $\max f'>a$, (1) cannot have a smooth solution in general. More precisely, *if $f'([\alpha,\beta])\subseteq(a,b)$ for some interval $[\alpha,\beta]\subseteq(0,1)$ and $f|_{(\alpha,\beta)}$ is not analytic, then* (1) *cannot have a solution with continuous partial derivative with respect to x.*

To see this, assume that u_x is continuous on $[\alpha,\beta]\times[0,T]$ and define $v(x,t):=u(x,T-t)$. Since $u_x(\cdot,0)=f'$ we have, for small enough T, $u_x([\alpha,\beta],[0,T])\subseteq(a,b)$. Hence v, restricted to the rectangle $[\alpha,\beta]\times[0,T]$, is a solution of

$$v_t=m_2v_{xx},$$
$$v(\alpha,t)=u(\alpha,T-t),\qquad v(\beta,t)=u(\beta,T-t),$$
$$v(x,0)=u(x,T),$$

and the smoothing property of the heat equation implies that $v(\cdot,T)=u(\cdot,0)=f$ is analytic on (α,β), which is contrary to the hypothesis.

In fact the above argument shows that, in general, u_x *cannot be piecewise continuous with respect to a finite partition of* $[0, 1] \times [0, T]$. This fact is also supported by numerical computations which we did jointly with C. de Boor. Approximations $u_x^n(\cdot, t)$ to $u_x(\cdot, t)$ oscillate in intervals where ϕ' is negative. This phenomenon has been independently observed by G. Strang and M. Abdel-Naby **[SA]**.

We use the following *notation* for the norms of the spaces C, L_p, C^α, $\alpha < 1$:

$$\|\psi\|_\infty = \sup_{\xi \in \Omega} |\psi(\xi)|, \qquad \|\psi\|_p = \left(\int_\Omega |\psi(\xi)|^p \, d\xi\right)^{1/p},$$
$$\|\psi\|_\alpha = \|\psi\|_\infty + \sup_{\xi, \xi' \in \Omega} |\psi(\xi) - \psi(\xi')| / |\xi - \xi'|^\alpha.$$

Unless explicitly specified the domain Ω will be clear from the context, e.g., in the Theorem below the norms are taken on $\Omega = [0, 1] \times [0, T]$.

THEOREM. *For* ϕ *as described by* (2) *and any* $f \in C^2[0, 1]$ *with* $\max f' > a$, $f'(0) = f'(1) = 0$, *there exists* $T > 0$ *such that problem* (1) *has infinitely many solutions* u *satisfying the equation* $u_t = \phi(u_x)_x$ *on* $[0, 1] \times [0, T]$ *in the sense of* L_2. *Each such solution satisfies the estimates* $\|u\|_\alpha$, $\|u_x\|_\infty$, $\|u_t\|_2 \leqslant c$, *where* T, c, α *depend on* ϕ *and* f. *Moreover, we have* $u_x([0, 1], (0, T]) \cap (a, b) = \varnothing$.

The last conclusion reflects the qualitative behavior of numerical solutions to problem (1) which has been observed in numerical experiments. It also shows that *our family of solutions does not depend on the values of* ϕ *in the interval* (a, b). In fact $\phi_{|(a,b)}$ could be defined arbitrarily. Also note that the solutions are slightly smoother than predicted by the a priori estimates mentioned earlier.

We think that the Theorem should extend to a more general class of smooth nonmonotone constitutive functions ϕ. Our technique of proof requires a linear relation between the monotone increasing parts of ϕ (cf. relation (3) below). Also one might think that imposing an additional condition (in analogy to hyperbolic conservation laws) leads to a (unique?) solution with special properties.

Before beginning with the proof of the Theorem let us choose a convenient normalization for the piecewise linear functions ϕ. The change of variables $u(x, t) = U(px, qt)$ transforms (1) into the equation

$$U_t = \Phi(U_x)_x$$

with $\Phi(s) = q^{-1} p \phi(ps)$. From this one can easily check that we may without loss of generality assume that $m_1 = 1$ and $\phi(a) + \phi(b) = 2$. If we define d by $\phi(1 + d) = \phi(1) = 1$, then, with this normalization, ϕ is completely determined by the three parameters a, b, d (cf. figure (2) where this normalization has already been chosen).

Crucial for the existence proof is the relation

$$\phi(s + A + Bs) = \phi(s), \qquad 2 - a \leqslant s \leqslant a, \tag{3}$$

where, to be precise,

$$A = \frac{ad + a + b - 2 - 2d}{a - 1}, \quad B = \frac{2 + d - a - b}{a - 1} > -1, \quad A + B = d > 0.$$

The following pathological feature of problem (1) is implied by identity (3). *For any function* $\chi\colon [0,1] \to \{0,1\}$ *and* $c \in [2-a, a]$,

$$u(x,t) = \int_0^x (c + \chi(y)(A + Bc))\, dy$$

is a solution of $0 = \phi(u_x)_x$, because by (3), $\phi(u_x) = c$.

In proving the Theorem we first consider in §1 the special case $B = 0$ which simplifies the analysis and illustrates the basic idea behind the proof of the general case done in §2. In the Appendix we state a regularity result for a linear parabolic equation needed for our arguments.

1. The case $B = 0$. For $B = 0$ the normalized ϕ is of the form

$$(2') \qquad \phi(s) = \begin{cases} s, & s \leqslant a, \\ s - d, & s \geqslant b, \end{cases}$$

and relation (3) becomes particularly simple:

$$(3') \qquad \phi(s + d) = \phi(s), \qquad 2 - a \leqslant s \leqslant a.$$

Numerical computations indicate the existence of solutions u with $u_x([0,1],(0,T]) \cap (a,b) = \varnothing$. This suggests splitting u into a smooth and an oscillating part, $u = v + w$. In view of (3′) we choose w of the form

$$(4) \qquad w(x,t) = d\int_0^x \chi(y,t)\, dy$$

with $\chi\colon [0,1]^2 \to \{0,1\}$, i.e. $w_x = d\chi$ assumes only the two values 0 and d:

$$(4.1) \qquad w_x \in \{0, d\}.$$

If $v_x \approx 1$, $\phi(v_x + w_x)_x = v_{xx}$, i.e. the oscillations of w_x are not recognized by the right-hand side of (1). This is the reason for the existence of solutions corresponding to initial data f with $f'([0,1]) \cap (a,b) \neq \varnothing$. The function χ, and hence w, will depend only on f and be constructed so that the resulting equation for v is as regular as possible. To this end, and for reasons that will become apparent in the proof of Proposition 1, we require that w satisfy

(4.2) $w_t \in L_\infty$;

(4.3) $w(x,0) = h(x)$, where

$$h(0) = 0 \quad \text{and} \quad h'(x) := \begin{cases} 0, & f'(x) \leqslant 1, \\ f'(x) - 1, & 1 \leqslant f'(x) \leqslant 1 + d, \\ d, & 1 + d \leqslant f'(x); \end{cases}$$

(4.4) *for all* $\varepsilon > 0$ *there exists* $T > 0$ *such that for* $t \leqslant T$,

$$\{f' \leqslant 1 - \varepsilon\} \subseteq \{\chi(\cdot, t) = 0\} \quad \textit{and} \quad \{f' \geqslant 1 + d + \varepsilon\} \subseteq \{\chi(\cdot, t) = 1\}.$$

Here we used the notation $\{\psi > y\} := \{x\colon \psi(x) > y\}$. Condition (4.4) means that the oscillations of χ are essentially restricted to a neighborhood of the set $\{x\colon f'(x) \in (1, 1 + d)\} \times [0, T]$.

Assuming the existence of a function w satisfying (4.1)–(4.4) we now construct a solution for problem (1).

PROPOSITION 1. *Let w of the form* (4) *satisfy* (4.1)–(4.4) *and define v as the solution of the problem*

$$(5)\qquad v_t = v_{xx} - w_t,\quad v_x(0,t) = v_x(1,t) = 0,\quad v(x,0) = g(x),$$

$$\text{where } g(0) = f(0) \text{ and } g'(x) := \begin{cases} f'(x), & f'(x) \leqslant 1, \\ 1, & 1 \leqslant f'(x) \leqslant 1 + d, \\ f'(x) - d, & 1 + d \leqslant f'(x). \end{cases}$$

Then there exists $T > 0$ such that $u = v + w$ is a solution of (1) *satisfying the regularity assertions of the Theorem.*

PROOF. From the definition of the initial values g, h for v, w, (4.4) and (5), it follows that u satisfies the boundary and initial conditions. Also note that g'', $h'' \in L_\infty$. This follows from the continuity of g', h' and $f \in C^2[0, 1]$.

In view of (5), the equation $u_t = \phi(u_x)_x$ is equivalent to

$$(*)\qquad \phi(v_x + w_x)_x = v_{xx}.$$

Since g'', $w_t \in L_\infty$, we have by Theorem A (cf. Appendix), applied to problem (5), that $v_x \in C^\alpha$. Therefore, for small t,

$$\{v_x(\cdot, t) \geqslant 1 + \varepsilon\} \subseteq \{g' \geqslant 1 + \varepsilon/2\} = \{f' \geqslant 1 + d + \varepsilon/2\}.$$

From (2′) and (4.4) it follows that for $x \in \{v_x(\cdot, t) > 1 + \varepsilon\}$,

$$\phi(v_x(x,t) + w_x(x,t)) = \phi(v_x(x,t) + d) = v_x(x,t).$$

We argue similarly if $x \in \{v_x(\cdot, t) < 1 - \varepsilon\}$. Finally, for $x \in \{|v_x(\cdot, t) - 1| < 2\varepsilon\}$ with $\varepsilon < (a - 1)/2$, we apply (3′) and (4.1) to complete the proof of (*).

The regularity assertions for u stated in the Theorem are consequences of (4.1), (4.2) and Theorem A. □

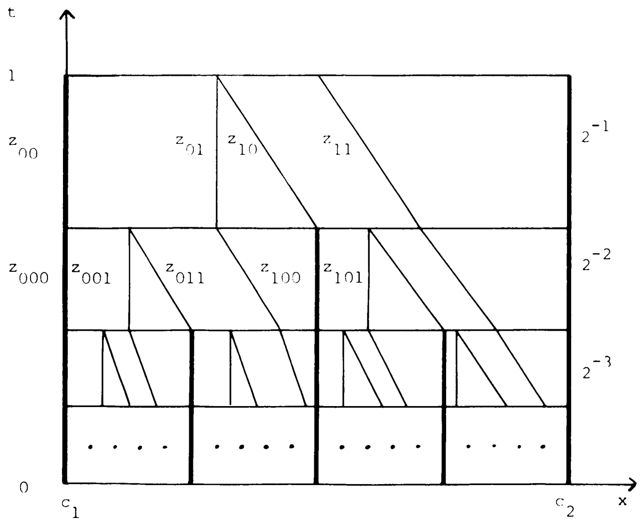
It remains to construct a function χ so that $w = d\int\chi$ satisfies (4.1)–(4.4). The difficulty lies in satisfying the initial condition $w(\cdot, 0) = h$ while $w_x = d\chi \in \{0, d\}$ and $w_t \in L_\infty$. We shall construct a piecewise linear function w with discontinuity pattern as indicated by figure (6) that interpolates $h(x_{jk})$ at the points $x_{jk} = c_1 + j \cdot 2^{-k}(c_2 - c_1)$, $j = 0, \dots, 2^k$, $k = 0, 1, \dots,$ where $[c_1, c_2]$ is an arbitrary interval containing supp h'. A modification of this construction will be used for the proof of the general piecewise linear case in §2.

In the case $A = 0$, considered in this section, it is possible to construct a function w satisfying (4.1)–(4.4) with a discontinuity pattern independent of the initial data. This has been observed by G. Strang and we also include his construction (cf. p. 306) which further illustrates the lack of uniqueness for problem (1).

We now describe the first mentioned construction and introduce some notation for the discontinuity pattern which will be useful in §2. Let $[c_1, c_2] \subseteq [0, 1]$ be any interval containing supp h' and consider the following infinite partition $\Pi(c_1, c_2, h)$

of $[c_1, c_2] \times [0, 1]$:

(6)



where we denote by z_r, $r = r_1 r_2 \cdots r_{|r|}$, $r_\nu \in \{0, 1\}$, the lines

$$t \to x = z_r(t): [2^{-|r|+1}, 2^{-|r|+2}] \to [0, 1].$$

The endpoints of these lines are denoted by $(\underline{z}_r, 2^{-|r|+1})$, $(\bar{z}_r, 2^{-|r|+2})$, respectively. We write rs, $s \in \{0, 1\}$, for $r_1 r_2 \cdots r_{|r|} s$. Whenever it is convenient we interpret r as the dual number $\sum_{\nu=1}^{|r|} r_\nu 2^{\nu-1}$, e.g. we write $r1 = r0 + 1$, etc. However, since we do not ignore leading zeros in the sequence r, different r's may correspond to the same number. As indicated by figure (6) we have

(7.1) $$\cdots \leq z_r \leq z_{r+1} \leq \cdots,$$

(7.2) $$\underline{z}_{r0} = \bar{z}_{r00} = z_{r00} = c_1 + r2^{-|r|}(c_2 - c_1),$$

(7.3) $$\underline{z}_{r1} = \bar{z}_{r11},$$

(7.4) $$\bar{z}_{r01} = \bar{z}_{r10} \in [\underline{z}_{r0}, \underline{z}_{r1}].$$

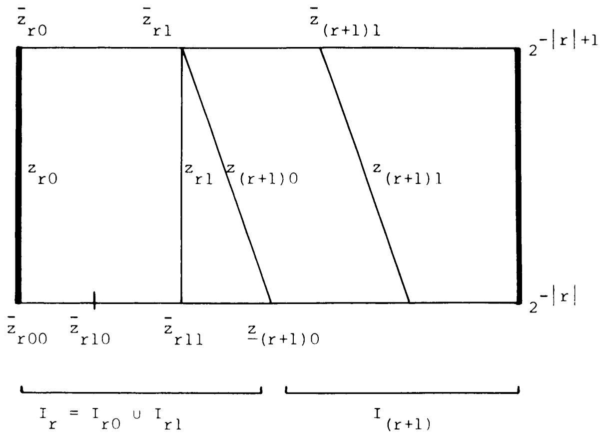
With this notation we define χ by

(8) $$\chi(x, t) = \begin{cases} 1, & z_{r0}(t) < x \leq z_{r1}(t),\ 2^{-|r|} \leq t \leq 2^{-|r|+1}, \\ 0, & \text{otherwise.} \end{cases}$$

We now choose the points $\bar{z}_{r1}$ which determine the partition Π so that w interpolates h on the vertical lines z_{r00} (bold lines in figure (6)); more precisely

(9) $$w(\underline{z}_{r0}, t) = h(\underline{z}_{r0}), \qquad t \leq 2^{-|r|}.$$

Clearly, this implies $lim_{t\to 0}\, w(\cdot\,, t) = h$, i.e. (4.3). Consider a typical subrectangle of the partition Π:



We define $\bar{z}_{r1}$ by

$$(10)\qquad d(\bar{z}_{r1} - \bar{z}_{r0}) = \int_{I_r} h'(y)\, dy, \qquad I_r := [\underline{z}_{r0}, \underline{z}_{(r+1)0}],$$

and note that $\bar{z}_{r0}$ is implicitly determined by (7.2) and (7.4).

Since $0 \leqslant h' \leqslant d$ and $I_{r0} \cup I_{r1} = I_r$ we have

$$\bar{z}_{r11} - \bar{z}_{r00} = d^{-1}\left(\int_{I_{r0}} h' + \int_{I_{r1}} h'\right) \leqslant \underline{z}_{(r+1)0} - \underline{z}_{r0},$$

i.e. $\underline{z}_{r1} = \bar{z}_{r11} \leqslant \bar{z}_{(r+1)00} = \underline{z}_{(r+1)0}$, which is consistent with (7.1). Moreover, we see that

$$d(\underline{z}_{r1} - \underline{z}_{r0}) = d(\bar{z}_{r11} - \bar{z}_{r00}) = \int_{I_r} h' = d(\bar{z}_{r1} - \bar{z}_{r0}),$$

i.e. the lines z_{r0}, z_{r1} are parallel. Therefore we have for $t \in [2^{-|r|-1}, 2^{-|r|}]$,

$$\int_{I_r} w_x(y, t)\, dy = d(\bar{z}_{r11} - \bar{z}_{r00}) = \int_{I_r} h',$$

which implies (9). Note that we have equality in (7.1) in either one of the following cases:

$$(11)\qquad \begin{aligned} z_{r0} &= z_{r1}, && \text{iff } h'(I_r) = 0,\\ z_{r01} &= z_{r10}, && \text{iff } h'(I_{r0}) = d,\\ z_{r11} &= z_{(r+1)00}, && \text{iff } h'(I_r) = d. \end{aligned}$$

We already saw that w, defined by (4) and implicitly by (10), satisfies (4.1) and (4.3). From (4) and figure (6) it is clear that w is continuous and therefore it is sufficient to compute w_t on the rectangles $[0, 1] \times (2^{-j}, 2^{-j+1})$. For $2^{-j} < t < 2^{-j+1}$ we have

$$(12)\qquad w(x, t) = \sum_{|r|=j} d\big((x - z_{r0}(t))_+ - (x - z_{r1}(t))_+\big),$$

(13) $$w_t(x,t)=\sum_{|r|=j} d\left(z'_{r1}(x-z_{r1}(t))^0_+ - z'_{r0}(x-z_{r0}(t))^0_+\right).$$

By (7.1),

$$\bar z_{r00}=z_{r00}\leqslant z_{r01}\leqslant z_{r10}\leqslant z_{r11}\leqslant \bar z_{(r+1)00},$$

with $\bar z_{(r+1)00}-\bar z_{r00}=2^{-|r|}(c_2-c_1)$, which implies

(7.5) $$0\leqslant z'_{r0}=z'_{r1}\leqslant 2.$$

This, together with (13), shows $\|w_t\|_\infty\leqslant 2$.

To prove (4.4), assume that $f'(x)\leqslant 1-\varepsilon$. By definition of g' we have $f'(x)=g'(x)$, $h'(x)=0$ in this case. Since $\operatorname{supp} h'\subset\{f'\geqslant 1\}$, the continuity of f' implies $\operatorname{dist}(x,\operatorname{supp} h')\geqslant\delta(\varepsilon)$. From (4), (8) and (11) we see that $w_x(x,t)=0$ for $t\leqslant\delta/2$. The second assertion of (4.4) is proved by a similar argument. □

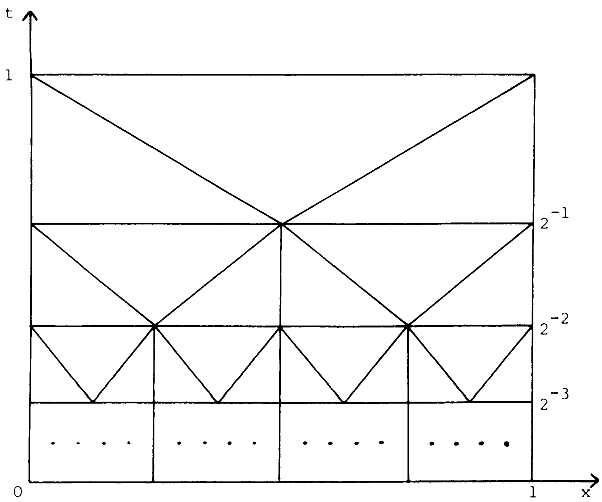
Nonuniqueness. Clearly, properties (4.1)–(4.4) do not determine w uniquely. For the construction of $\Pi(c_1,c_2,h)$, we could choose any interval $[c_1,c_2]$ that contains $\operatorname{supp} h'$. Also we may perturb the points $\bar z_{r1}$ which determine the discontinuity pattern of $w_x=d\chi$. The discontinuities distinguish w from the smooth part v of the solution u and therefore we get a continuum of solutions for problem (1).

An alternative construction of w. G. Strang pointed out to us a nice construction of a function w satisfying (4.1)–(4.4) which has a fixed discontinuity pattern, independent of the initial data h. His idea can be briefly described as follows.

Given h with $h(0)=0$ and $0\leqslant h'(x)\leqslant d$, $x\in[0,1]$, there exist continuous piecewise linear functions s_k with

$$s'_{k|((j-1)2^{-k},j2^{-k})}\in\{0,d\},\qquad j=1,\dots,2^k,$$
$$\|h-s_k\|_\infty\leqslant d\cdot 2^{-k}.$$

Consider the following partition of $[0,1]^2$,



and define w as the piecewise linear continuous function with respect to this partition that agrees with s_k on the lines $(x,2^{-k})$, $x\in[0,1]$. It can easily be checked that w satisfies (4.1)–(4.4). Moreover, the construction is not unique. Any scaling $t\mapsto\alpha t$ gives a different w with the same properties.

2. The general case. As in the previous section we construct a solution u of the form $u = v + w$ where v is smooth and w is a function with oscillating derivative with respect to x. In view of (3) we choose w of the form

$$w(v, x, t) = \int_0^x \chi(v, y, t)(A + Bv_x(y, t))\, dy, \tag{4'}$$

with $\chi\colon [0, 1] \times [0, T] \to \{0, 1\}$. To obtain a sufficiently regular equation for v it seems to be necessary to let χ, i.e. the discontinuity pattern of the solution u, depend on v. An appropriate choice of χ will yield $w \in C^\alpha$ and $w_t = B\chi v_t + \psi$ with $\psi \in L_\infty$. This choice of χ, and hence of w, leads to an equation for v of the form (cf. Proposition 2)

$$v_t + B\chi(v, \cdot)v_t + \psi(v, \cdot) = v_{xx}.$$

To solve it, we have to study the structure of the partition corresponding to χ and the dependence of w, χ, ψ on v.

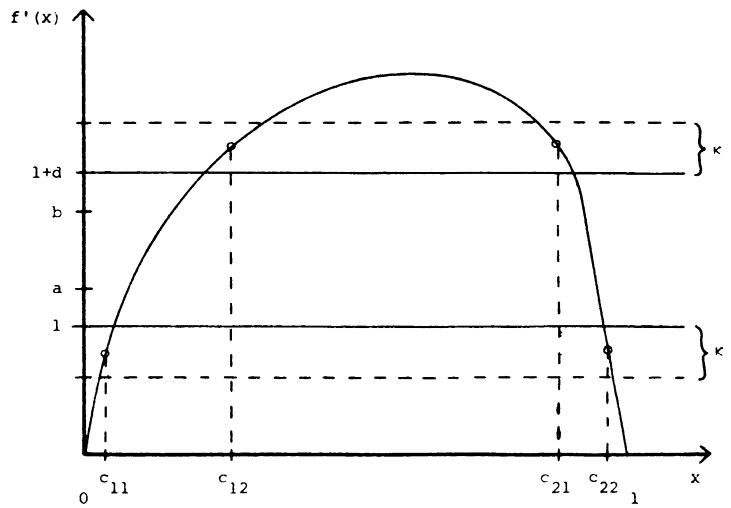
We define the initial values for v and w by $g(0) := f(0)$, $h(0) := 0$ and

$$\begin{aligned} g'(x) &:= \begin{cases} f'(x), & f'(x) \leqslant 1, \\ 1, & 1 \leqslant f'(x) \leqslant 1 + d, \\ (f'(x) - A)/(1 + B), & 1 + d \leqslant f'(x), \end{cases} \\ h'(x) &:= \begin{cases} 0, & f'(x) \leqslant 1, \\ f'(x) - 1, & 1 \leqslant f'(x) \leqslant 1 + d, \\ (A + Bf'(x))/(1 + B), & 1 + d \leqslant f'(x). \end{cases} \end{aligned} \tag{14}$$

Using $A + B = d$ (cf. (3)) one can easily check that g', h' are continuous, and from $f \in C^2[0, 1]$ it follows that $g'', h'' \in L_\infty$.

In order not to complicate the proof of the Theorem by unessential technical details we assume that for the initial function f the set $\{1 \leqslant f' \leqslant 1 + d\}$ consists of at most two intervals (cf. figure (14′) below). In general we would have to carry out the constructions described below separately for a finite number of intervals of a suitably chosen partition of [0, 1].

(14′)



The partition $\Pi(v)$. Throughout this and the following paragraph we fix a function v, with $v, v_x \in C^\alpha$, that satisfies the initial condition $v(\cdot, 0) = g$ with g defined by (14), (14′).

Since $a > 1, A + B = d > 0, B > -1$ (cf. (3)) there is a constant κ with $\kappa < (a - 1)$ such that

$$A + B[1-\kappa, 1+\kappa] \subseteq (\kappa, d + |B|\kappa].$$

Let us assume that *max* $g' > 1 + \kappa$ because this is the slightly more complicated case for the construction of the partion $\Pi(v)$. We can find intervals $I_j = [c_{j1}, c_{j2}]$, $j = 1, 2$, and a constant $\kappa'(g')$ with $0 < \kappa' < \kappa$ and $2\kappa' < c_{21} - c_{12}$ so that

$$\text{(15)} \qquad \begin{aligned} g'\big([c_{j1} - \kappa', c_{j2} + \kappa']\big) &\subseteq (1-\kappa, 1+\kappa), \qquad j = 1, 2, \\ g'([c_{12}, c_{21}]) &\subseteq (1 + \kappa', \infty). \end{aligned}$$

Moreover, we may assume that c_{12}, c_{21} are of the form $c_{11} + \nu 2^{-N}(c_{22} - c_{11})$, where $N = N(f')$ is sufficiently large. Since $v_x \in C^\alpha$ and $v_x(\cdot, 0) = g$ *there exists* $T(g', \|v_x\|_\alpha) > 0$ *such that, for* $t \leqslant T$, (15) *remains valid with* g' *replaced by* $v_x(\cdot, t)$. This implies in particular

$$\text{(16)} \qquad A + Bv_x(x,t) \in (\kappa, d + |B|\kappa), \qquad (x,t) \in I_j' \times [0, T], j = 1, 2,$$

where $I_j' = [c_{j1} - \kappa', c_{j2} + \kappa']$, a fact we shall frequently use in the sequel.

LEMMA 1. *There exists* $T > 0$ *such that for* $t \leqslant T$ *the equation*

$$\text{(17)} \qquad Ax + Bv(x,t) = Ay + Bg(y)$$

defines two one-to-one maps

$$\Lambda_j \colon I_j \times [0,T] \to \Omega_j \subseteq I_j' \times [0,T] \colon (y,t) \to (x,t), \qquad j = 1, 2,$$

which are strictly monotone increasing in the first coordinate.

PROOF. Since $v \in C^\alpha$ and $v(\cdot, 0) = g$ there exists $T(g, \|v\|_\alpha, \kappa') > 0$ such that for $t \leqslant T$,

$$AI_j' + Bv(I_j', t) \supseteq AI_j + Bg(I_j),$$

which implies that (17) can be solved for $x \in I_j'$ if $y \in I_j$. To complete the proof, we note that by (16) both sides of (17) are strictly increasing functions of their arguments x, y, respectively. □

For the definition of $\Pi(v)$ described below it is convenient to define a single map

$$\Lambda \colon [c_{11}, c_{22}] \times [0,T] \to [0,1] \times [0,T]$$

which agrees with Λ_j on $I_j \times [0, T]$. To this end we set

$$ZL(t) = \Lambda_1(c_{12}, t), \qquad ZR(t) = \Lambda_2(c_{21}, t),$$

i.e. ZL, ZR denote the right and left boundaries of Ω_1, Ω_2, respectively, and define

$$\text{(18)} \qquad \Lambda(y,t) = \begin{cases} \Lambda_j(y,t), & (y,t) \in I_j \times [0,T], \\ \dfrac{c_{21} - y}{c_{21} - c_{12}} ZL(t) + \dfrac{y - c_{12}}{c_{21} - c_{12}} ZR(t), & c_{12} < y < c_{21}. \end{cases}$$

As the individual maps Λ_j, Λ is one-to-one and strictly monotone increasing in the first coordinate. We cannot use (17) to define Λ, since for $x \in (c_{12}, c_{21})$ the function $Ax + Bv(x, t)$ need not be monotone increasing,and hence (17) may not be uniquely solvable for x.

The partition $\Pi(v)$ is a perturbation of the partition $\Pi(c_{11}, c_{22}, h_0)$ described in §1, where the function h_0 is defined by

$$h_0(0) = 0, \qquad h_0' := dh'/(A + Bg')$$

with h given by (14). Note that $\Pi(c_{11}, c_{22}, h_0)$ refers to the partition constructed from h_0 rather than from h. We keep the notation introduced in §1, in particular we denote by z_r the lines determining the partition $\Pi(c_{11}, c_{22}, h_0)$. We define $\Pi(v)$ as the image of $\Pi(c_{11}, c_{22}, h_0)$ under the map Λ, i.e. we replace the lines z_r, for $|r| > 2 - \log_2 T$, by the curves

$$(19) \qquad z_r(v, \cdot): [2^{-|r|+1}, 2^{-|r|+2}] \to [0, 1], \quad (z_r(v, t), t) = \Lambda(z_r(t), t).$$

Since the points c_{12}, c_{21} had been chosen of the form $c_{11} + \nu 2^{-N}(c_{22} - c_{11})$, the right and left boundaries of the rectangles $I_j \times [2^{-|r|+1}, 2^{-|r|+2}]$, $j = 1, 2$, agree with lines from the partition $\Pi(c_{11}, c_{22}, h_0)$. Therefore, for the definition of one particular curve $z_r(v, \cdot)$, Λ either coincides with Λ_j or is given by the second formula in (18). We denote by Ξ the set of all r for which the first possibility applies, i.e.

$$\Xi = \{r: z_r(t) \in I_1 \cup I_2, t \in [2^{-|r|+1}, 2^{-|r|+2}]\}.$$

By Lemma 1 and definition (18) of Λ, the partition $\Pi(v)$ has the same structure as $\Pi(c_{11}, c_{22}, h_0)$. By this we mean that

$$(7.1') \qquad \cdots \leq z_r(v, \cdot) \leq z_{r+1}(v, \cdot) \leq \cdots,$$

$$(7.2') \qquad \underline{z}_{r0}(v) = \bar{z}_{r00}(v),$$

$$(7.3') \qquad \underline{z}_{r1}(v) = \bar{z}_{r11}(v),$$

$$(7.4') \qquad \bar{z}_{r01}(v) = \bar{z}_{r10}(v) \in [\underline{z}_{r0}(v), \underline{z}_{r1}(v)],$$

where $\bar{z}_r(v)$, $\underline{z}_r(v)$ denote the upper and lower endpoints of the curves $z_r(v, \cdot)$. Note that we have equality in (7.1′) iff we have equality in (7.1) (cf. (11)). Since $h_0'([c_{12}, c_{21}]) = d$, this implies that

$$z_{r1}(v, \cdot) = z_{(r+1)0}(v, \cdot), \qquad r1 \notin \Xi.$$

The following lemma shows that for $t \to 0$ the partition $\Pi(v)$ "converges" to $\Pi(c_{11}, c_{22}, h_0)$.

LEMMA 2. $|z_r(v, t) - z_r(t)| \leq ct^\alpha$.

Here and in the sequel c denotes various positive constants which may depend on f, α, $\|v\|_\alpha$, $\|v_x\|_\alpha$. Also, we shall always assume $|r| > 2 - \log_2 T$ so that the curves $z_r(v, \cdot)$ are well defined.

PROOF. We may assume $r \in \Xi$. Writing (17) in the form

$$A(x - y) + B(v(x, t) - v(y, t)) + B(v(y, t) - g(y)) = 0,$$

we obtain the estimate

$$|(x-y)(A+Bv_x(\xi,t))|\leqslant ct^\alpha$$

and the lemma follows from (16). □

LEMMA 3. $\|z_r(v,\cdot)\|_\alpha\leqslant c$, *uniformly in* r.

PROOF. We may assume $r\in\Xi$, and to simplify notation we set $x=z_r(v,t)$, $x'=z_r(v,t')$, $y=z_r(t)$, $y'=z_r(t')$. From (17) we see that

$$A(x-x')+B(v(x,t)-v(x',t))+B(v(x',t)-v(x',t'))$$
$$=A(y-y')+B(g(y)-g(y')).$$

Writing $v(x,t)-v(x',t)=v_x(\xi,t)(x-x')$, using (16) and the estimate $|g(y)-g(y')|\leqslant c|z_r'||t-t'|$ finishes the proof. □

The functions $\chi(v,\cdot)$ *and* $w(v,\cdot)$. Denote by R_r, R_r' the deformed rectangles

$$R_r:=\{(x,t):z_{r0}(v,t)<x\leqslant z_{r1}(v,t),\ t\in[2^{-|r|},2^{-|r|+1}]\},$$
$$R_r':=\{(x,t):z_{r1}(v,t)<x\leqslant z_{(r+1)0}(v,t),\ t\in[2^{-|r|},2^{-|r|+1}]\}.$$

Corresponding to the partition Π, constructed in the previous paragraph, we define the function χ by

$$\chi(v,x,t)=\begin{cases}1, & (x,t)\in R_r,\\ 0, & \text{otherwise}.\end{cases} \tag{8'}$$

From the remark following (7.4′) we see that

$$\chi(v,x,t)=1,\quad ZL(t)\leqslant x\leqslant ZR(t), \tag{20}$$

i.e. χ does not depend on the particular form of the curves $z_r(v,\cdot)$ for $r\notin\Xi$. We gave an explicit definition for these curves merely because then we do not have to treat each of the intervals I_1, I_2 separately.

Substituting (8′) into definition (4′) for w we obtain

$$w(v,x,t)=\begin{cases}\sum_{s\leqslant r}A(z_{s1}(t)-z_{s0}(t))+B(g(z_{s1}(t))-g(z_{s0}(t))), \\ \qquad\qquad\qquad\qquad \text{if } (x,t)\in R_r',\\ \sum_{s<r}A(z_{s1}(t)-z_{s0}(t))+B(g(z_{s1}(t))-g(z_{s0}(t))) \\ \qquad +Ax+Bv(x,t)-Az_{r0}(t)-Bg(z_{r0}(t)),\\ \qquad\qquad\qquad\qquad \text{if } (x,t)\in R_r.\end{cases} \tag{12'}$$

This follows from (17), (18). E.g., if $(x,t)\in R_r'$ and $0\leqslant x\leqslant ZL(t)$, we have

$$\begin{aligned}w(v,x,t)&=\sum_{s\leqslant r}\int_{z_{s0}}^{z_{s1}}(A+Bv_x(y,t))\,dy\\ &=\sum A(z_{s1}(v,t)-z_{s0}(v,t))+B(v(z_{s1}(v,t),t)-v(z_{s0}(v,t),t))\\ &=\sum A(z_{s1}(t)-z_{s0}(t))+B(g(z_{s1}(t))-g(z_{s0}(t))).\end{aligned}$$

We argue similarly if $(x, t) \in R_r$ and $0 \leqslant x \leqslant ZL(t)$. If $ZL(t) \leqslant z_{r1}(v, t) \leqslant ZR(t)$, we have $z_{r1} = z_{(r+1)0}$, which implies $R'_r = \varnothing$. Therefore, if $x \in (ZL(t), ZR(t))$, $x \in R_r$,

$$w(v, x, t) = w(v, ZL(t), t) + \int_{ZL(t)}^{x} (A + Bv_x(y, t))\, dy$$

agrees with (12′) because the terms $Az_{s1} + Bg(z_{s1})$ and $Az_{(s+1)0} + Bg(z_{(s+1)0})$ cancel. Finally, if $x \geqslant ZR(t)$, we argue similarly as for $x \leqslant ZL(t)$.

If $B = 0$ one can check that $h_0 = h$, $z_r(v, \cdot) = z_r$, $\Pi(v) = \Pi(c_{11}, c_{22}, h)$, $w_x = A\chi = d\chi$, which shows that our definition is consistent with the special case treated in §1.

LEMMA 4. $lim_{t \to 0} \| w(v, \cdot, t) - h \|_\infty = 0$.

PROOF. Since $| w_x | \leqslant | A | + | B | | v_x | \leqslant c$, it is sufficient to check the convergence for sufficiently many points. We shall show that

$$w(v, z_{r00}(v, t), t) \to h(z_{r00}(v, t)), \qquad | r | \to \infty,$$

uniformly in $t \in [2^{-|r|-1}, 2^{-|r|}]$. In view of Lemma 2 we can replace $h(z_{r00}(v, t))$ by $h(z_{r00}(t)) = h(\underline{z}_{r0})$. From (12′) we see that

$$w(v, z_{r00}(v, t), t) = \sum_{s<r0} A(z_{s1}(t) - z_{s0}(t)) + B(g(z_{s1}(t)) - g(z_{s0}(t))).$$

By definition of the partition $\Pi(c_{11}, c_{22}, h_0)$ (cf. (10)) we have

$$z_{s1}(t) - z_{s0}(t) = \bar{z}_{s1} - \bar{z}_{s0} = d^{-1} \int_{I_s} h'_0.$$

Using this, $A + B = d$ and the mean value theorem twice, we obtain

$$\begin{aligned} \sum_{s<r0} \cdots &= \sum (A + Bg'(\xi_s)) d^{-1} \int_{I_s} h'_0 \\ &= \sum (\bar{z}_{(s+1)00} - \bar{z}_{s00})(h'_0(\zeta_s)(A + Bg'(\xi_s))/d). \end{aligned}$$

This can be interpreted as a Riemann sum for $\int_{[c_{11}, \underline{z}_{r0}]} h'_0(A + Bg')/d = \int_{[c_{11}, \underline{z}_{r0}]} h'$, which proves the lemma since $h'' \in L_\infty$. □

From (12′) we can formally compute w_t (cf. p. 313 for a proof) which is given by

$$w_t(v, x, t) = B\chi(v, x, t) v_t(x, t) + \psi(v, x, t) \tag{13′}$$

where

$$\psi(v, x, t) = \begin{cases} \sum_{s \leqslant r} B(g'(z_{s1}(t)) - g'(z_{s0}(t))) z'_{s0}, & (x, t) \in R'_r, \\ \sum_{s<r} B(g'(z_{s1}(t)) - g'(z_{s0}(t))) z'_{s0} \\ \qquad - (A + Bg'(z_{r0}(t))) z'_{r0}, & (x, t) \in R_r. \end{cases} \tag{13″}$$

Since $g'' \in L_\infty$ we have $\psi \in L_\infty$ and, if $v_t \in L_2$, $w_t \in L_2$.

LEMMA 5. $\| w(v, \cdot) \|_{\alpha^2} \leqslant c$.

PROOF. Since $w_x \in L_\infty$ it is sufficient to prove the Hölder continuity with respect to t. In estimating $w(v, x, t) - w(v, x, t')$ let us first assume that $t, t' \in [2^{-j}, 2^{-j+1}]$. We consider two cases:

1. For some $r, |r| = j, (x, t), (x, t') \in R_r$. In this case it follows from (12′) that

$$\begin{aligned}
&|w(v, x, t) - w(v, x, t')| \\
&\quad \leqslant \Big| \sum_{s<r} B\big(g(z_{s1}(t)) - g(z_{s1}(t'))\big) + B\big(g(z_{s0}(t')) - g(z_{s0}(t))\big)\Big| \\
&\qquad + |B|\,|v(x, t) - v(x, t')| + A\,|z_{r0}(t) - z_{r0}(t')| \\
&\qquad + |B|\,|g(z_{r0}(t)) - g(z_{r0}(t'))| \\
&\quad \leqslant |t - t'| \sum |B|\,|g'(\xi_s)z'_{s1} - g'(\zeta_s)z'_{s0}| + c\,|t - t'|^\alpha + c\,|t - t'| \\
&\quad \leqslant c\,|t - t'|^\alpha.
\end{aligned}$$

We obtain the same estimate if $(x, t), (x, t') \in R'_r$.

2. Now consider the case when the points $(x, t), (x, t')$ do not lie in a common deformed rectangle R_r or R'_r. It follows from Lemma 2 that at most $c2^{j\alpha}$ of the curves $z_{r\nu}(v, \cdot)$ can intersect the segment $(x, [t, t'])$. Therefore we can find $t = t_0 < t_1 < \cdots < t_N = t'$, $N \leqslant c2^{j\alpha}$, such that for each pair $(x, t_\nu), (x, t_{\nu+1})$ either one of the previous cases applies. Using Hölder's inequality, we obtain

$$|w(v, x, t) - w(v, x, t')| \leqslant c \sum_{\nu=1}^{N} |t_\nu - t_{\nu-1}|^\alpha \leqslant N^{1-\alpha}|t' - t|^\alpha \leqslant c\,|t' - t|^{\alpha^2},$$

where for the last inequality we used $N \leqslant c\,|t' - t|^{-\alpha}$.

The general case follows now easily. For $t \in [2^{-j}, 2^{-j+1}]$, $t' \in [2^{-k}, 2^{-k+1}]$, $j < k$, we obtain, using the previous estimates,

$$\begin{aligned}
|w(v, x, t) - w(v, x, t')| &\leqslant c\left(|t - 2^{-j}|^{\alpha^2} + |2^{-k+1} - t'|^{\alpha^2} + \sum_{\nu=j+1}^{k-1} 2^{-\nu\alpha^2}\right) \\
&\leqslant c\,|t - t'|^{\alpha^2}. \quad \square
\end{aligned}$$

The dependence of Π, χ, w, ψ *on* v. Throughout this paragraph, which is the final preparation for the proof of the Theorem, we shall restrict v to the set

(21) $$K := \{v : \|v\|_\alpha, \|v_x\|_\alpha, \|v_t\|_2 \leqslant c,\ v(\cdot, 0) = g\}.$$

We note that the constants in the previous lemmas, in particular κ, κ' and T, can be chosen uniformly with respect to $v \in K$.

LEMMA 6. *The following maps are continuous.*

(22.1) $$v \to z_r(v, \cdot) : (K, \|\ \|_\infty) \to C([2^{-|r|+1}, 2^{-|r|+2}]),$$

(22.2) $$v \to \chi(v, \cdot) : (K, \|\ \|_\infty) \to L_2([0, 1] \times [0, T]),$$

(22.3) $$v \to \psi(v, \cdot) : (K, \|\ \|_\infty) \to L_2([0, 1] \times [0, T]),$$

(22.4) $$v \to w(v, \cdot) : (K, \|\ \|_\alpha, \|\partial_x\ \|_\alpha) \to C([0, 1] \times [0, T]).$$

PROOF OF (22.1). We may assume $r \in \Xi$. For $v, v' \in K$ and the corresponding curves $x = z_r(v, \cdot)$, $x' = z_r(v', \cdot)$, we obtain from (17):

$$A(x - x') + B(v(x,t) - v(x',t)) + B(v(x',t) - v'(x',t)) = 0,$$

which implies $|x - x'| \leqslant c\|v - v'\|_\infty$ because of (16). □

PROOF OF (22.2). Since $\chi \in \{0, 1\}$ we have

$$\|\chi(v, \cdot) - \chi(v', \cdot)\|_2^2 \leqslant \varepsilon + \|\chi(v, \cdot) - \chi(v', \cdot)\|^2_{2,[0,1]\times[\varepsilon,T]}.$$

For any $\varepsilon > 0$ only finitely many of the curves $z_r(v, \cdot)$ overlap the rectangle $[0, 1] \times [\varepsilon, T]$. This observation finishes the proof in view of definition (8′) of χ and (22.1). □

We skip the proof of (22.3) which is a slight extension of this argument.

PROOF OF (22.4). Since for $v \in K$, $w(v, \cdot, 0) = h$, and $w \in C^\alpha$, it is sufficient to prove the continuity into $C([0, 1] \times [\varepsilon, T])$, any fixed $\varepsilon > 0$. Skipping most of the subscripts we write $w(v, x, t) - w(v', x, t)$ in the form (cf. (4′))

$$\int_0^x (A(\chi(v) - \chi(v')) + B(\chi(v) - \chi(v'))v_x + B\chi(v')(v_x - v'_x)).$$

From the definition of χ and (22.1) we see that

$$\|\chi(v, \cdot, t) - \chi(v', \cdot, t)\|_2 \to 0, \quad \text{as } \|v - v'\|_\infty \to 0,$$

uniformly for $t \geqslant \varepsilon$, which finishes the proof. □

With the aid of Lemma 6 we can now justify the formal computation of w_t.

PROOF OF (13′). Let us first assume that v_t is continuous. From (17) we see that this implies $z_r(v, \cdot) \in C^1([2^{-|r|+1}, 2^{-|r|+2}])$. Hence R_r, R'_r have a piecewise C^1 boundary, and since w is continuous we may compute w_t separately on these sets. In this case (13′) is a direct consequence of definition (12′) of w.

To finish the proof for $v_t \in L_2$ we choose a smooth approximating sequence $v^n \in K$, $v^n \to v$, and note that by Lemma 6, $w(v^n, \cdot) \to w(v, \cdot)$, $\chi(v^n, \cdot)v_t \to \chi(v, \cdot)v_t^n$, $\psi(v^n, \cdot) \to \psi(v, \cdot)$ in L_2. □

The proof of the Theorem is based on the following proposition.

PROPOSITION 2. *There exists* $T > 0$ *such that the problem*

$$(5')\qquad \begin{aligned} &v_t + B\chi(v, \cdot)v_t + \psi(v, \cdot) = v_{xx} \\ &v_x(0, t) = v_x(1, t) = 0, \qquad v(\cdot, 0) = g, \end{aligned}$$

has a solution on the rectangle $[0, 1] \times [0, T]$ *satisfying* $\|v\|_\alpha$, $\|v_x\|_\alpha$, $\|v_t\|_2$, $\|v_{xx}\|_2 \leqslant c$, *where* c, T *and* α *depend on* g.

PROOF. We solve (5′) by iterating in the form

$$(*)\qquad v_t^n + B\chi(v^{n-1}, \cdot)v_t^n + \psi(v^{n-1}, \cdot) = v_{xx}^n$$

with boundary and initial conditions as in (5′). Since $B > -1$ we have $0 < \min(1, 1/(1 + B)) \leqslant 1/(1 + B\chi)$, $\|\psi\|_\infty \leqslant c$ and we can apply Theorem A to get uniform bounds for $\|v^n\|_\alpha$, $\|v_x^n\|_\alpha$, $\|v_t^n\|_2$, $\|v_{xx}^n\|_2$, i.e. all iterates stay in a set of the form (21). We choose T small enough, so that for all n, $\chi(v^{n-1}, \cdot)$, $\psi(v^{n-1}, \cdot)$ are well defined, i.e. $T = T(K)$. By compactness we can select for $\alpha' < \alpha$ a subsequence, again denoted by v^n, for which $(*)$ holds and $v^n \to v$, $v_x^n \to v_x$ in $C^{\alpha'}$, $v_t^n \to v_t$,

$v_{xx}^n \to v_{xx}$ weakly in L_2. To pass to the limit in (*), i.e. to show that v is a solution of (5′), we note that by Lemma 6 all terms in (*) converge weakly in L_2 to the corresponding expressions in (5′). E.g. we have

$$\lim \int \chi(v^{n-1}, \cdot) v_t^n \phi = \lim \int (\chi(v, \cdot)\phi) v_t^n + \lim \int (\chi(v^{n-1}, \cdot) - \chi(v, \cdot)) v_t^n \phi$$
$$= \int \chi(v, \cdot) \phi v_t,$$

where for the last step we used (22.2) and $\| v_t^n \phi \|_2 \leqslant c$. □

PROOF OF THE THEOREM. Let v be a solution of (5′). We claim that $u = v + w$ solves problem (1).

From $\operatorname{supp} w \subseteq [c_{11} - \kappa, c_{22} + \kappa] \times [0, T]$, (5′) and Lemma 4 we see that u satisfies the correct boundary and initial conditions. By (13′) and (5′) we have $v_t + w_t = v_{xx}$. Therefore it remains to show that

$$(*) \qquad v_x = \phi(v_x + w_x) = \phi(v_x + \chi(A + Bv_x)).$$

Set $\Omega := \{(x, t): ZL(t) < x < ZR(t), t \in [0, T]\}$. By the continuity of v_x we have, for sufficiently small $T > 0$,

$$v_x([0,1] \times [0, T] \setminus \Omega) \subseteq (-\infty, 1 + \kappa)$$

and

$$v_x(\Omega \cup \Omega_1 \cup \Omega_2) \subseteq (1 - \kappa, \infty).$$

Moreover, χ is equal to 1 on Ω and equal to 0 on $[0,1] \times [0, T] \setminus \{\Omega \cup \Omega_1 \cup \Omega_2\}$. Therefore, (*) follows from (2) and (3) which imply

$$\phi(s) = s, \qquad s \leqslant a,$$
$$\phi(s + A + Bs) = s, \qquad s \geqslant 2 - a.$$

The regularity assertions of the Theorem are consequences of Lemma 5 and Proposition 2. □

Nonuniqueness. By definition (8′) of χ, w_x is discontinuous across the curves z_r. These discontinuities distinguish w from v, the smooth part of the solution u. One way of obtaining a continuum of different choices for w is to perturb the partition $\Pi(c_{11}, c_{22}, h_0)$ and, hence, $\Pi(v)$, e.g., as follows. In the construction of these partitions we replace the intervals $[2^{-|r|}, 2^{-|r|+1}]$ by $[\lambda 2^{-|r|}, \lambda 2^{-|r|+1}]$ with $\lambda = 1$.

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Appendix.

THEOREM A. *For* $a, f \in L_\infty([0,1] \times \mathbf{R}_+)$ *with* $0 < a_0 \leqslant a$, *and* $g \in C^\beta([0,1])$, $\beta > 1$, *with* $g'(0) = g'(1) = 0$, *the problem*

$$\text{(A)} \qquad \begin{aligned} &u_t = a u_{xx} + f, \qquad (x, t) \in [0,1] \times \mathbf{R}_+, \\ &u_x(0, t) = u_x(1, t) = 0, \qquad u(\cdot, 0) = g \end{aligned}$$

has a unique solution satisfying $\|u\|_\alpha$, $\|u_x\|_\alpha$, $\|u_{xx}\|_2 \leqslant c$ *where* $\alpha > 0$ *and* c *depend on* a_0, $\|a\|_\infty$, $\|f\|_\infty$, $\|g\|_\beta$, β.

There seems to be no convenient reference for this result. Therefore we include an argument deducing it from results in [**L**].

Differentiating (A) with respect to x we obtain, with $v = u_x$,

$$\text{(A}'\text{)} \qquad v_t = (av_x)_x + f_x, \qquad v(0,t) = v(1,t) = 0, \qquad v(\cdot, t) = g'.$$

This is an equation of the form (1.1) in [**L**, p. 134]. From Theorems 4.1, 4.2, 7.1, 10.1 in [**L**, pp. 133 − 210] with $n = 1$, $\nu = \alpha_0$, $\mu = \|a\|_\infty$, $q = r = \infty$, $\mu_1 = \max(\|a\|_\infty, \|f\|_\infty)$, it follows that (A′) has a unique solution satisfying $\|v\|_\alpha$, $\|v_x\|_2 \leqslant c$ where $\alpha > 0$ and c depend on the quantities above. (For our purposes there is no point in distinguishing between the Hölder continuity with respect to x and t.)

With v the solution of (A′) we set

$$(1) \qquad u(x,t) = h(t) + \int_0^x v(y,t)\,dy.$$

Formally substituting this into (A) we find that

$$h'(t) + \int_0^x v_t(y,t)\,dy = (av_x)(x,t) + f(x,t)$$

and, therefore,

$$(2) \quad h(t) = g(0) + \int_0^t ((av_x)(x,\tau) + f(x,\tau))\,d\tau - \int_0^x (v(y,t) - v(y,0))\,dy.$$

For $\phi \in L_2$ we interpret $\int \phi(x,\tau)\,d\tau$ as $\lim \int \phi^n(x,\tau)\,d\tau$, where ϕ^n is a smooth approximating sequence, using the fact that the map $\phi \to \int_0^\cdot \phi(\cdot,\tau)\,d\tau\colon L_2 \to L_2$ is continuous.

To justify the definition of h we have to show that the right-hand side of (2) does not depend on x. To this end let ϕ be a test function with $\operatorname{supp}\phi \subseteq (0,1) \times (0,T)$ and define η by $\eta_t = \phi$, $\eta(\cdot, T) = 0$. Integrating by parts we obtain

$$\iint h\eta_{tx} = -\iint ((av_x)(x,t) + f(x,t))\eta_x(x,t)\,dx\,dt$$
$$+ \iint (v(x,t) - v(x,0))\eta_t(x,t)\,dx\,dt = 0,$$

i.e. $h_x = 0$. For the last equality we have used the fact that v is a weak solution of (A′) (cf. [**L**, p. 136]) and $-\iint v(x,0)\eta_t(x,t)\,dx\,dt = \int v(x,0)\eta(x,0)\,dx$. From the definition of h one can now easily check that u is a solution of (A).

To see that h, and hence u, is Hölder continuous we write

$$|h(t') - h(t)| = \left|\int_0^1 (h(t') - h(t))\,dx\right|$$
$$\leqslant \left|\int_t^{t'} \int_0^1 ((av_x)(x,\tau) + f(x,\tau))\,dx\,d\tau\right| + \int_0^1 \int_0^x |v(y,t') - v(y,t)|\,dy\,dx$$
$$\leqslant \| |av_x| + |f| \|_2 |t' - t|^{1/2} + c|t' - t|^\alpha. \quad \square$$

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